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Improving Efficiency of Survey Sample Procedures through Order Statistics

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SUMMARY

The classical estimator of the mean of a finite population is the sample mean. Two new estimators are introduced based on the order statistics of a simple random sample. It is shown that these estimators are considerably more efficient than the sample mean for all sample sizes small or large over a very wide family of symmetric distributions. These results are extended to stratified sampling. Skew distributions will be considered in a future paper.

Key words : Survey sampling, Order statistics, MML estimators, Robustness, Symmetric distributions, Cost function, Stratification.

1. *Introduction*

Let $y_1, y_2, ..., y_n$ be a simple random sample (drawn without replacement) from a finite population Π_N consisting of the elements $Y_1, Y_2, ..., Y_N$. Let

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i \qquad (1.1)$$

be the mean of Π_N. The classical estimator of $\bar{y}$ is the sample mean

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \qquad (1.2)$$

which is very widely used in sample survey procedures.

There are essentially two approaches in sample survey. One approach is to consider the finite population Π_N as having been selected at random from a hypothetical population (referred to as a super-population); the observed simple random sample $y_1, y_2, \ldots, y_n$ being a random sample from Π_N, is also a random sample from the super-population. The second approach does not

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make any reference to a super-population in which case the empirical verification entails repeatedly sampling the same objective finite population Π_N. For philosophic foundations of these two approaches, the reader is referred to Cochran [5] and Godambe [11].

The sample mean $\bar{y}$ is an efficient estimator of $\bar{Y}$, with either of the two approaches, if the underlying distribution is normal. However, non-normal distributions are more prevalent in practice; see Elveback, Guillier and Kerting [9], Hüber [12] and Tiku, Tan and Balakrishnan [24]. We consider two other estimators. The first one is due to Tiku ([14], [18]) and has already been introduced in sample survey (Tiku [19]). The second one is due to Tiku and Suresh [23] and is introduced in sample survey in this paper. The efficiencies of both these estimators are evaluated in the framework of super-population models. It is shown that both are considerably more efficient than the sample mean $\bar{y}$. It is illustrated that these efficiency properties also hold in the framework of finite population models. Extensions to stratified sampling are discussed.

2. *Super-Population Model*

Most classical statistical procedures are based on the assumption of normality. This assumption, however, is unrealistic from a practical point of view. To quote R.C. Geary (*Biometrika*, 1947): "Normality is a myth; there never was, and never will be, a normal distribution". This might be an overstatement, but the fact is that non-normal distributions are more prevalent in practice, and to assume normality instead might lead to erroneous statistical inferences. We will, therefore, consider a super-population model representing a wide class of location-scale symmetric distributions. This important class consists of the distributions

$$f(y;p) = \frac{1}{\sigma\sqrt{k}\,\beta(\frac{1}{2}, p-\frac{1}{2})}\left\{1+\frac{(y-\mu)^2}{k\sigma^2}\right\}^{-p}, \quad -\infty < y < \infty \qquad (2.1)$$

where $k = 2p-3$ and $p \geq 2$. Note that $E(y) = \mu$ and $V(y) = \sigma^2$. For $1 < p < 2$, k is taken to be equal to 1 in which case μ and σ are simply the location and scale parameters, respectively. For $p = \infty$, (2.1) reduces to a normal distribution $N(\mu, \sigma^2)$. For $p = 1$, (2.1) represents a Cauchy distribution. In fact, $t = \sqrt{v}\,(y-\mu)/\sigma\sqrt{k}$ has Student's distribution with $v = 2p-1$ degrees of freedom. The kurtosis μ_4/μ_2^2 (Pearson measure of non-normality) of the family (2.1) ranges between 3 (when $p=\infty$) and infinity (when $p \leq 2.5$).

3. *Estimators of Location*

In the framework of super-population model let $y_1, y_2, ..., y_n$ be a simple random sample from Π_N.

Classical estimator : The classical estimator of $\overline{Y}_N = \overline{Y}$ is the sample mean $\overline{y} = \overline{y}_n$,

$$\overline{y}_n = \frac{1}{n} \sum_{i=1}^{n} y_i \tag{3.1}$$

Let $\overline{Y}_{N-n}$ be the mean of the remaining $N - n$ observations in Π_N. Now

$$\overline{Y}_N = \frac{n}{N} \overline{y}_n + \left(1 - \frac{n}{N}\right) \overline{Y}_{N-n}, \tag{3.2}$$

Realize that $\overline{y}_n$ and $\overline{Y}_{N-n}$ are unconditionally independent (Fuller [10]). Therefore,

$$\begin{aligned} E(\overline{y}_n - \overline{Y}_N)^2 &= \left(1 - \frac{n}{N}\right)^2 E\{(\overline{y}_n - \mu) - (\overline{Y}_{N-n} - \mu)\}^2 \\ &= \left(1 - \frac{n}{N}\right)^2 \left\{\frac{\sigma^2}{n} + \frac{\sigma^2}{N-n}\right\} = \frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right) \end{aligned} \tag{3.3}$$

which is, of course, a well-known result. The classical estimator of σ^2 is

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \overline{y})^2$$

Note that $E(\overline{y}_n - \overline{Y}_N) = 0$ since $E(\overline{y}_n) = \mu$ and $E(\overline{Y}_{N-n}) = \mu$.

Robust estimator : One of the most popular robust estimator of μ is due to Tiku ([14], [18]). This estimator is based on the symmetric type II censored sample

$$y_{(r+1)} \leq y_{(r+2)} \leq \ldots \leq y_{(n-r)} \tag{3.4}$$

obtained by arranging the observations $y_1, y_2, ..., y_n$ in ascending order of magnitude and censoring the r smallest and largest observations. The underlying super-population model for (3.4) is presumed to be normal $N(\mu, \sigma^2)$. This is supported by the premise that non-normality essentially comes from the tails (Tiku *et al.* [24], pp. 22-25) and once the extreme observations (representing the tails) are censored (given zero weights), there is hardly any difference between a normal and a non-normal sample.

The robust estimator of μ is (Tiku ([14], [18]); Tiku *et al.* [24]),

$$\hat{\mu}_c = \frac{\sum_{r+1}^{n-r} y_{(i)} + r\beta\,(y_{(r+1)} + y_{(n-r)})}{m} \tag{3.5}$$

with
$$V(\hat{\mu}_c) = (1'V\,1)\,\sigma^2 / m^2 \tag{3.6}$$

where $1' = (0, \ldots, 0, 1 + r\beta, 1, \ldots, 1, 1 + r\beta, 0, \ldots, 0)$ is a $n \times 1$ vector of constant coefficients and V is the $n \times n$ variance-covariance matrix of the standardized order statistics $z_{(i)} = \{ y_{(i)} - \mu \} / \sigma \; (i = 1, 2, \ldots, n)$. These variances and covariances are available for the family (2.1); for $p = 1$ in Barnett [2] and Vaughan [27], for $p = 1.5$ in Vaughan [25], for $p = 2\,(.5)\,10$ in Tiku and Kumra [21], and for $p = \infty$ in *Biometrika Tables Vol. II*. Note that $E(\hat{\mu}_c) = \mu$; this follows from the fact that $E\{z_{(i)}\} = -E\{z_{(n-i+1)}\}$ $(i = 1, 2, .., n)$. For $p = \infty$ (normal distribution), $V(\hat{\mu}_c) \cong \sigma^2 / m$ (Tiku [16]). The robust estimator of σ is

$$\hat{\sigma}_c = \{ B + \sqrt{(B^2 + 4AC)} \} / 2\sqrt{\{A(A-1)\}} \tag{3.7}$$

In (3.5) - (3.7), $m = n - 2r + 2r\beta$, $A = n - 2r$, $B = r\alpha\{ y_{(n-r)} - y_{(r+1)} \}$ and

$$C = \sum_{r+1}^{n-r} y_{(i)}^2 + r\beta\left(y_{(r+1)}^2 + y_{(n-r)}^2\right) - m\,\hat{\mu}_c^2 \tag{3.8}$$

The constants α and β are obtained from the equations

$$\beta = -\phi(t)\{ t - \phi(t)/q \}/q \text{ and } \alpha = \{\phi(t)/q\} - \beta t, \quad (q = r/n) \tag{3.9}$$

where $\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ and t is determined by the equation $\int_{-\infty}^{t} \phi(z)\,dz = 1 - q$. It is easy to evaluate the values of α and β. A table of their values is, however, given in Tiku *et al.* ([24], pp. 74-75) for convenience.

For symmetric population, r is chosen to be the integer value $r = [0.5 + 0.1n]$ for reasons given in Tiku [18]. For symmetric population with nonexistent variance (Cauchy, for example), the efficiency of $\hat{\mu}_c$ is further enhanced by taking $r = [0.5 + 0.3n]$; see Tiku [18].

Since $\hat{\mu}_c$ and $\overline{Y}_{N-n}$ are unconditionally independent

$$E(\hat{\mu}_c - \overline{Y}_N)^2 = V(\hat{\mu}_c) + \frac{n}{N}\frac{\sigma^2}{n} - 2\frac{n}{N}\,\mathrm{Cov}(\hat{\mu}_c, \bar{y}_n) \tag{3.10}$$

Note that $E(\hat{\mu}_c - \overline{Y}_N) = 0$

Since $\bar{y}_n = \sum_{i=1}^{n} y_{(i)} / n$

$$\mathrm{Cov}(\hat{\mu}_c, \bar{y}_n) = (\mathbf{1}' \mathbf{V} \mathbf{w})\, \sigma^2 / m \tag{3.11}$$

where $\mathbf{1}'$ and $\mathbf{V}$ are the same as in (3.6) and $\mathbf{w}'$ is the $n \times 1$ vector $(1/n, 1/n, \ldots, 1/n)$.

If we put r equal to zero in (3.4) - (3.8), then $\hat{\mu}_c$ and $\hat{\sigma}_c^2$ reduce to the sample mean $\bar{y}$ and the sample variance s^2, respectively.

MML estimator : Consider the super-population model (2.1). In the first place we assume that p is known. Let

$$y_{(1)} \le y_{(2)} \le \ldots \le y_{(n)} \tag{3.12}$$

be the order statistics of the random sample $y_1, y_2, \ldots, y_n$. The likelihood function of these order statistics is

$$L \propto \left(\frac{1}{\sigma}\right)^n \prod_{i=1}^{n} \left[1 + \frac{(y_{(i)} - \mu)^2}{k\sigma^2}\right]^{-p} \tag{3.13}$$

The ML (maximum likelihood) estimators of μ and σ are solutions of the equations

$$\frac{\partial \ln L}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial \ln L}{\partial \sigma} = 0 \tag{3.14}$$

These equations are intractable and have no explicit solutions. Solving them by iteration is problematic. For small values of p, particularly, one can encounter multiple roots, slow convergence or convergence to wrong values; see Barnett [3], Tiku and Suresh [23] and Vaughan [26]. MML estimators are obtained by replacing L by an asymptotically equivalent function L^* such that

$$\lim_{n \to \infty} \frac{1}{n} \,|L - L^*| = 0 \tag{3.15}$$

For details about this methodology, see Tiku ([14], [15], [17]), Tiku *et al.* [24], Tiku and Stewart [22] and Tiku and Suresh [23]. The MML estimators of μ and σ are thus obtained :

$$\hat{\mu}_n = \left\{ \sum_{i=1}^{n} \beta_i y_{(i)} \right\} \Big/ M, \qquad \left(M = \sum_{i=1}^{n} \beta_i \right) \tag{3.16}$$

and

$$\hat{\sigma}_n = \{ B + \sqrt{(B^2 + 4nC)} \} / 2 \sqrt{n(n-1)} \tag{3.17}$$

$\sigma_n^2 \cong C/(n-1)$ since B^2 is very small as compared to C. Here

$$B = \frac{2p}{k} \sum_{i=1}^{n} \alpha_i y_{(i)} \text{ and } C = \frac{2p}{k} \sum_{i=1}^{n} \beta_i \{y_{(i)} - \hat{\mu}_n\}^2 = \frac{2p}{k} \left\{ \sum_{i=1}^{n} \beta_i y_{(i)}^2 - M\hat{\mu}_n^2 \right\}$$

and the constant coefficients α_i and β_i are obtained from the equations $t_{(i)} = E(z_{(i)})$:

$$\alpha_i = \frac{(2/k)\, t_{(i)}^3}{[1 + (1/k)\, t_{(i)}^2]^2} \text{ and } \quad \beta_i = \frac{1 - (1/k)\, t_{(i)}^2}{[1 + (1/k)\, t_{(i)}^2]^2}, \quad (i = 1, 2, \ldots, n) \tag{3.18}$$

Note the umbrella-ordering of both α_i and β_i. In fact, α_i is of order $o\,(t_{(i)}^{-1})$ and β_i is of order $o\,(t_{(i)}^{-2})$. Consequently, the extreme observations in (3.16) – (3.17) get small weights. That makes the estimators above robust to outliers. For $p = \infty$ (normal distribution), $\alpha_i = 0$ and $\beta_i = 1$ $(i = 1, 2, \ldots, n)$. Asymptotically, $\hat{\mu}_n$ and $\hat{\sigma}_n$ are minimum variance bound estimators. The minimum variance bound for estimating μ is

$$\text{MVB}(\mu) = \left\{ \left(p - \frac{3}{2}\right)(p+1) \Big/ np \left(p - \frac{1}{2}\right) \right\} \sigma^2 \quad \text{for } p \geq 2 \tag{3.19}$$

$$= \left\{ (p+1) \Big/ 2np \left(p - \frac{1}{2}\right) \right\} \sigma^2 \qquad \text{for } 1 \leq p < 2$$

In fact, the variance $V(\hat{\mu}_n)$ is very close to (3.19) for all $n \geq 5$ (Tiku and Suresh [23], Table 1; Vaughan [26]).

Since $t_{(i)} = -t_{(n-i+1)}$, $\hat{\mu}_n$ is an unbiased estimator of μ. Consequently, $E(\hat{\mu}_n - \overline{Y}_N) = 0$ and since $\hat{\mu}_n$ and $\overline{Y}_{N-n}$ are unconditionally independent

$$E(\hat{\mu}_n - \overline{Y}_n)^2 = V(\hat{\mu}_n) + \frac{n}{N}\frac{\sigma^2}{n} - 2\frac{n}{N}\,\mathrm{Cov}(\hat{\mu}_n, \bar{y}_n) \tag{3.20}$$

The covariance term is evaluated exactly the same way as in (3.10).

4. *Efficiency of the Estimators*

Equations (3.3), (3.10) and (3.20) give the following results of MSE (mean square error) of $\bar{y}_n$, $\hat{\mu}_c$ and $\hat{\mu}_n$:

$$E(\bar{y}_n - \overline{Y}_N)^2 \le E(\hat{\mu}_c - \overline{Y}_N)^2 \quad \text{and} \quad E(\bar{y}_n - \overline{Y}_N)^2 \le E(\hat{\mu}_n - \overline{Y}_N)^2$$

if $\sigma^2 / n \le B_1$ and $\sigma^2 / n \le B_2$, respectively. The bounds B_1 and B_2 are given by

$$B_1 = V(\hat{\mu}_c) + 2\frac{n}{N}\left\{\frac{\sigma^2}{n} - \mathrm{Cov}(\hat{\mu}_c, \bar{y}_n)\right\} \tag{4.1}$$

and

$$B_2 = V(\hat{\mu}_n) + 2\frac{n}{N}\left\{\frac{\sigma^2}{n} - \mathrm{Cov}(\hat{\mu}_n, \bar{y}_n)\right\} \tag{4.2}$$

The exact values of the variances $V(\hat{\mu}_c)$ and $V(\hat{\mu}_n)$ and covariances $\mathrm{Cov}(\hat{\mu}_c, \bar{y}_n)$ and $\mathrm{Cov}(\hat{\mu}_n, \bar{y}_n)$ are given in Appendix. Using this Appendix, the exact values of B_1 and B_2 were calculated for the representative values $p = 2, 3.5, 5$ and ∞ and sample size $n = 6, 10$ and 20. For $p = 5$, (2.1) is indistinguishable from a logistic distribution since the two have the same first four moments (Pearson [13]; Tiku and Jones [20]). The values of σ^2 / n and B_1 and B_2 are given in Table 1 for the sample fraction $n / N = 0.1$ and 0.2. In most surveys n / N is less than or equal to 0.1 but, for a broader coverage, we included the value $n / N = 0.2$.

It is clear from Table 1 that $\hat{\mu}_c$ is more efficient than $\bar{y}_n$ except when $p = \infty$. The estimator $\hat{\mu}_n$ is more efficient than $\bar{y}_n$ except when $p = \infty$ in which case the two estimators are identical and have, therefore, the same efficiency. All this is because σ^2 / n is greater than B_1 except when $p = \infty$ and is greater than B_2 always except when $p = \infty$ in which case σ^2 / n is equal to B_2. Note that for $p < 2$, the sample mean $\bar{y}_n$ has zero efficiency since its MSE is infinite.

Table 1. Exact values of σ^2 / n, B_1 and B_2

n/N	n=6			n=10			n=20		
	1/n	B_1/σ^2	B_2/σ^2	1/n	B_1/σ^2	B_2/σ^2	1/n	B_1/σ^2	B_2/σ^2
					p=2				
0.1	0.1667	0.1152	0.1112	0.1000	0.0713	0.0648	0.0500	0.0346	0.0314
.2		.1268	.1254		.0773	.0746		.0378	.0365
					p=3.5				
0.1	0.1667	0.1569	0.1547	0.1000	0.0938	0.0909	0.0500	0.0464	0.0448
.2		.1605	.1569		.0954	.0928		.0473	.0460
					p=5				
0.1	0.1667	0.1639	0.1619	0.1000	0.0981	0.0962	0.0500	0.0487	0.0477
.2		.1660	.1627		.0991	.0969		.0492	.0482
					p=∞				
0.1	0.1667	0.1812	0.1667	0.1000	0.1035	0.1000	0.0500	0.0518	0.0500
.2		.1836	.1667		.1035	.1000		.0518	.0500

The exact values of the MSE of the estimators $\bar{y}_n$, $\hat{\mu}_c$ and $\hat{\mu}_n$ are also calculated for n = 6, 10 and 20 and p = 1.5, 2, 3.5, 5 and ∞ and given in Table 2

Table 2. Exact values of the mean square errors of the estimators, sample size n=10

p	n/N	MSE $(\bar{y}_n)/\sigma^2$	MSE $(\hat{\mu}_c)/\sigma^2$	MSE $(\hat{\mu}_n)/\sigma^2$
1.5	0.1	∞	0.1255*	0.0849
	.2	∞	.0977*	.0705
2	0.1	0.0900	0.0613	0.0548
	.2	.0800	.0573	.0546
3.5	0.1	0.0900	0.0838	0.0809
	.2	.0800	.0754	.0728
5	0.1	0.0900	0.0882	0.0861
	.2	.0800	.0791	.0769
∞	0.1	0.0900	0.0935	0.0900
	.2	.0800	.0835	.0800

* For r = [0.5 + 0.3n], the MSE are 0.0868 and 0.0718 for n/N = 0.1 and 0.2, respectively.

for n = 10 only, for brevity; the values are similar for n = 6 and 20, and higher values of n. For $p < 2$ the sample mean $\bar{y}_n$ has infinite MSE and is, therefore, a very poor estimator as compared to $\hat{\mu}_c$ and $\hat{\mu}_n$. For all values of $p \geq 2$, $\bar{y}_n$ has larger MSE than both $\hat{\mu}_c$ and $\hat{\mu}_n$ unless p = ∞ (normal distribution) in which case $\bar{y}_n$ has somewhat smaller MSE than $\hat{\mu}_c$ but has the same MSE as $\hat{\mu}_n$. Clearly, $\hat{\mu}_n$ is the most efficient estimator and is considerably more efficient than $\bar{y}_n$. It may be noted, however, that $\hat{\mu}_c$ is an omnibus estimator: it does not depend on any particular value of p in (2.1) and can also be used for other models, e.g., when the sample contains outliers or the sample comes from a mixture of two normal distributions. (Tiku *et al.* [24], Section 5.8). This is not to say that $\hat{\mu}_n$ (and $\hat{\sigma}_n$) cannot be used in such situations. In fact $\hat{\mu}_n$ (and $\hat{\sigma}_n$) can be worked out for any location-scale distribution of the type $(1/\sigma)f[(y-\mu)/\sigma]$ and will surely be more efficient than $\hat{\mu}_c$. The expressions for $\hat{\mu}_n$ and $\hat{\sigma}_n$, however, will have to be worked out for each family and these expressions involve specialized tables of expected values of order statistics; see also Vaughan [26] for some important remarks and steps to be followed in calculating $\hat{\sigma}_n$.

Estimating the shape parameter :

In calculating $\hat{\mu}_n$ and $\hat{\sigma}_n$ we assumed that the shape-parameter p in (2.1) is known. In practice, there might be situations when p is not known. In such situations, one obtains a plausible value of p from a Q – Q plot (see Section 7) or estimates p by solving the equation

$$\partial \ln L / \partial p = 0 \qquad (4.3)$$

The solution of this equation is rounded to the nearest integer or half-integer since the expected values of order statistics for fractional value of 2p are not available. The parameter p in (2.1) is equated to this estimated value and the calculation of $\hat{\mu}_n$ and $\hat{\sigma}_n$ proceeds as before. The equations (4.3) has no explicit solution. Therefore, $\hat{p}$ has to be obtained iteratively. One starts with a guessed value $p = p_0$ and calculates $\hat{\mu}_n$ and $\hat{\sigma}_n$ from (3.16) - (3.18). These values are substituted in (4.3). If $\partial \ln L / \partial p$ is close to zero, then $\hat{\mu}_n$ and $\hat{\sigma}_n$ are the required estimates. If not, one repeats this process with a different value of p and continues till $\partial \ln L / \partial p$ gets close to zero. This iteration process is easy to carry out since $\hat{\mu}_n$ and $\hat{\sigma}_n$ are explicit functions of sample observations. Similar estimators have been developed by Tiku [15] and Cohen and Whitton ([7], [8]).

To evaluate the effect of estimating p on the MSE of $\hat{\mu}_n$, we did an extensive Monte Carlo simulation study. We were pleased to notice that the effect of estimating p as against not estimating p is to increase the MSE of $\hat{\mu}_n$ but only marginally. Thus $\hat{\mu}_n$ retains its superiority over $\hat{\mu}_c$ and $\bar{y}_n$ in terms of having smaller MSE. It must be said, however, that although $\hat{\mu}_c$ trails behind $\hat{\mu}_n$ in terms of efficiency but its simplicity is remarkable and the fact is that it is on the whole considerably more efficient than $\bar{y}_n$. It may also be noted that for near-normal populations, the linear convex combination estimator

$$\mu_n^* = (n / N)\, \bar{y}_n + (1 - n / N)\, \hat{\mu}_c \qquad (4.4)$$

has smaller MSE than $\hat{\mu}_c$; see Tiku [19] for details.

5. *Finite Population Model*

For finite population models, as said earlier, the elements Y_i $(i = 1, 2, \ldots, N)$ of Π_N are fixed. A sample y_i $(i = 1, 2, \ldots, n)$ consists of n of these elements chosen at random without replacement. The order statistics of this sample are substituted in (3.5) - (3.8) and (3.16) - (3.18) to obtain $\hat{\mu}_c$ and $\hat{\mu}_n$ (and $\hat{\sigma}_c$ and $\hat{\sigma}_n$).

It is well known that $E(\overline{y}_n) = \overline{Y}_N$ and $E(\overline{y}_n - \overline{Y}_N)^2 = (S^2/n)(1 - n/N)$, $S^2 = \sum_{i=1}^{N} (Y_i - \overline{Y}_N)^2/(N-1)$. The expected values of $\hat{\mu}_c$, $\hat{\mu}_n$ and their MSE are obtained exactly the same way as before, the only difference being in the expected values, variances and covariances of the order statistics $y_{(i)}$ $(i = 1, 2, ..., n)$ which are now obtained as follows :

The sampling distribution of $y_{(i)}$ is given by (Brownlee [4], Section 3.8; Wilks [28], pp. 243, 252) :

$$p\{y_{(i)} = Y_{(t)}\} = \frac{\binom{t-1}{i-1}\binom{N-t}{n-i}}{\binom{N}{n}} \quad p_{N,n,i}(t), \text{ say} \tag{5.1}$$

where $t = i, i+1, ..., N-n+i$

The joint sampling distribution of $y_{(i)}$ and $y_{(j)}$ is given by $(i \leq j)$

$$p\{y_{(i)} = Y_{(t_1)}, y_{(j)} = Y_{(t_2)}\} = \frac{\binom{t_1-1}{i-1}\binom{t_2-t_1-1}{j-i-1}\binom{N-t_2}{n-j}}{\binom{N}{n}}$$

$$= p_{N,n,i,j}(t_1, t_2) \tag{5.2}$$

where $t_1 = i, i+1, ..., N-n+i$ and $t_2 = j, j+1, ..., N-n+j$. Thus

$$E(y_{(i)}) = \sum_{t=i}^{N-n+i} Y_{(t)}\, p_{N,n,i}(t) \tag{5.3}$$

$$E(y_{(i)}^2) = \sum_{t=i}^{N-n+i} Y_{(t)}^2\, p_{N,n,i}(t) \tag{5.4}$$

and

$$E(y_{(i)}\, y_{(j)}) = \sum_{t_1=i}^{N-n+i} \sum_{t_2=j}^{N-n+j} Y_{(t_1)} Y_{(t_2)}\, p_{N,n,i,j}(t_1, t_2) \tag{5.5}$$

The expressions (5.3) - (5.5) can be computed from the ordered $Y_{(i)}$ $(i = 1, 2, ..., N)$ values in Π_N. As a check on these computations, if $Y_{(i)}$ $(i = 1, 2, ..., N)$ are replaced by the integers 1, 2, ..., N, respectively, then (Wilks [28], pp. 243, 252)

$$E(y_{(i)}) = \frac{i(N+1)}{n+1} \tag{5.6}$$

$$V(y_{(i)}) = E(y_{(i)}^2) - [E(y_{(i)})]^2 = \frac{i(N-n)(N+1)(n-i+1)}{(n+1)^2(n+1)} \tag{5.7}$$

and

$$\begin{aligned} \text{Cov}(y_{(i)}, y_{(j)}) &= E(y_{(i)} y_{(j)} - E(y_{(i)}) E(y_{(j)}) \\ &= \frac{i(n-j+1)(N-n)(N+1)}{(n+1)^2(n+2)} \end{aligned} \tag{5.8}$$

A further check on these values is that

$$E(\bar{y}_n) = w' \mu_{i:n} = \bar{Y} \text{ and } V(\bar{y}_n) = w' V w = (S^2/n)(1 - n/N) \tag{5.9}$$

where $\mu_{i:n} = E\{y_{(i)}\}$, $\sigma_{ij:n} = \text{Cov}(y_{(i)}, y_{(j)})$ and $V = (\sigma_{ij:n})$ $(i, j = 1, 2, ..., n)$ is the $n \times n$ variance-covariance matrix.

Efficiency of the estimators : Regarding Π_N as a fixed finite population, a random sample of size N is generated from the symmetric family (2.1); see also Tiku [19]. The exact values of the relative efficiencies calculated from (5.1) – (5.5) are given below for the representative values $n = 20$ and $n/N = 0.1$ and 0.2, $p = 1.5, 2.5, 5$ and ∞ :

$$E_1 = 100 \sqrt{\left\{ \frac{E(\bar{y}_n - \bar{Y}_N)^2}{E(\hat{\mu}_c - \bar{Y}_N)^2} \right\}} \text{ and } E_2 = 100 \sqrt{\left\{ \frac{E(\bar{y}_n - \bar{Y}_N)^2}{E(\hat{\mu}_n - \bar{Y}_N)^2} \right\}} \tag{5.10}$$

Relative efficiencies

	n/N = 0.1				n/N = 0.2			
p	1.5	2.5	5	∞	1.5	2.5	5	∞
E_1	302	114	102	96	113	103	101	95
E_2	360	123	105	100	128	108	103	100

It is clear that $\hat{\mu}_c$ and $\hat{\mu}_n$ are both considerably more efficient than $\bar{y}_n$ except when $p = \infty$ (normal distribution) in which case $\hat{\mu}_c$ is a little less efficient but $\hat{\mu}_n$ is as efficient as $\bar{y}_n$. For symmetric populations, $\hat{\mu}_c$ and

$\hat{\mu}_n$ are both unbiased estimators of $\overline{Y}_N$. Of course, $\bar{y}_n$ is an unbiased estimator of $\overline{Y}_N$ always. Note that the relative efficiencies have essentially the same magnitudes under finite-population as well as super-population models and do not depend on the size n of the simple random sample. We have not reproduced the values for large n for brevity.

Although it is reported above the relative efficiencies only for a small sample of size n = 20 but, it may be noted, that these results hold true for all values of n howsoever large, for fixed n/N.

6. *Standard Errors*

The following results on standard errors are true under both super-population as well as finite-population models :

The standard error of $\bar{y}_n$ is, for all population with finite mean and variance, given by

$$\pm \sqrt{\left\{ \frac{1}{n} \left(1 - \frac{n}{N} \right) s^2 \right\}} \tag{6.1}$$

This result is quite well known.

If the underlying population is normal $N(\mu, \sigma^2)$, then

$$\text{Cov}(\hat{\mu}_c, \bar{y}_n) = \frac{\sigma^2}{n} \tag{6.2}$$

This follows from the well-known result that the sum of each row (and each column) of the variance-covariance matrix of the order statistics of a random sample of size n from a normal population N(0, 1) is 1. Using this result, it immediately follows from (3.10) that for a normal population $N(\mu, \sigma^2)$ the standard error of $\hat{\mu}_c$ is given by

$$\pm \sqrt{\left\{ \frac{1}{m} \left(1 - \frac{n}{N} \right) \hat{\sigma}_c^2 \right\}} \tag{6.3}$$

since $V(\hat{\mu}_c) \cong \sigma^2 / m$ as said earlier. Tiku ([19], p. 2048) showed that (6.3), due to the fact that $V(\sqrt{m}\,\hat{\mu}_c / \hat{\sigma}_c) \cong 1$ for normal as well as symmetric non-normal populations, gives close approximations to the true standard error always.

The standard error of $\hat{\mu}_n$ for the family (2.1) is closely approximated by (equation 3.19)

$$\pm \sqrt{\left\{ \frac{\left(p - \frac{3}{2}\right)(p+1)}{np\left(p - \frac{1}{2}\right)} \left(1 - \frac{n}{N}\right) \hat{\sigma}_n^2 \right\}}, \quad (p \geq 2) \tag{6.4}$$

This is due to the fact that Cov $(\hat{\mu}_n, \bar{y}_n)$ is very close to V $(\hat{\mu}_n)$; see Table A. Thus (3.20) is closely approximated by

$$\left(1 - \frac{n}{N}\right) V(\hat{\mu}_n) + \frac{\sigma^2}{N} \left\{1 - \frac{n}{\sigma^2} V(\hat{\mu}_n)\right\} \tag{6.5}$$

Realizing that $0 < (n/\sigma^2) V(\hat{\mu}_n) \leq 1$, the standard error of $\hat{\mu}_n$ to order $O(N^{-1/2})$ is given by (6.4). An extensive Monte Carlo study confirmed the remarkable accuracy of (6.4). We do not reproduce the values for conciseness. For $p = \infty$ (normal distribution), (6.5) reduces to (6.1)

For large n and small n/N, all the three estimators $\bar{y}_n$, $\hat{\mu}_c$ and $\hat{\mu}_n$ are approximately normally distributed. The length of a confidence interval being determined by the magnitude of the standard error of an estimate, the confidence interval based on $\hat{\mu}_n$ will clearly be the shortest on the average, since $E(s) \cong \sigma$, $E(\hat{\sigma}_n) \cong \sigma$ and $E(\hat{\sigma}_c) \cong \sigma$ (for $p = \infty$).

Example : It is stated in Section 4 that the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n$ have very high precision and are fairly insensitive to moderate departures from the true value of p in (2.1). To illustrate this, consider the following $n = 10$ ordered observations :

8.4449 9.0782 9.4161 9.6709 9.8933 10.1067 10.3291 10.5839 10.9218 11.5551

These observations come from (2.1) exactly with $p = 3.5$, $\mu = 10$ and $\sigma = 1$, since they were obtained by adding 10 to each of the corresponding $t_{(i)}$ $(i = 1, 2, ..., 10)$ values.

The estimates calculated from (3.16) - (3.19) are $\hat{\mu} = 10$ and $\hat{\sigma} = 1.00$ which are exactly the population values. The sample mean and sample standard deviation are $\bar{y} = 10$ and $s = 0.91$.

To illustrate the robustness to moderate departures from the true value $p = 3.5$, assume a somewhat different value for p and compute the estimates from the data above. For $p = 3$, we get $\hat{\mu} = 10$ and $\hat{\sigma} = 1.03$. For $p = 4$, we get $\hat{\mu} = 10$ and $\hat{\sigma} = 0.98$. It is clear that the estimators $\hat{\mu}_n$ and $\hat{\sigma}_n$ are fairly robust numerically; see also Bian and Tiku [1].

7. *Sample Size Determination*

Often in practice one wants to pre-determine the sample size n such that the MSE of a subsequent estimator (of μ) does not exceed a given limit D. The estimators $\bar{y}_n$ and $\hat{\mu}_n$ are typically suited to solve such sample-size determination problems :

Equating the MSE of $\bar{y}_n$ to D, we get

$$n = \frac{\sigma^2}{(D + \sigma^2 / N)} \tag{7.1}$$

Equating the MSE of $\hat{\mu}_n$ to D, we get

$$n \cong \frac{\left(p - \frac{3}{2}\right)(p+1)}{p\left(p - \frac{1}{2}\right)} \frac{\sigma^2}{(D + \sigma^2 / N)} \tag{7.2}$$

Since $\left(p - \frac{3}{2}\right)(p+1) / p\left(p - \frac{1}{2}\right) < 1$, $\hat{\mu}_n$ requires a considerably smaller n than $\bar{y}_n$ to attain the same pre-determined MSE. Consequently, its use will lead to considerable saving in the cost.

Values of σ^2 and p in (7.1) - (7.2) are obtained from a training sample. The information about p is obtained from a Q - Q plot. That is, the order statistics of the training sample are plotted against the expected values $t_{(i)}$ $(i = 1, 2, .., n)$. The value of p which produces a straight line (or closest to such) is the required value of p. For large n (> 20, say) $t_{(i)}$ may be obtained from the equation

$$\frac{1}{\sqrt{k}\, \beta\left(\frac{1}{2}, p - \frac{1}{2}\right)} \int_{-\infty}^{t_{(i)}} \left\{1 + \frac{z^2}{k}\right\}^{-p} dz = \frac{i}{n+1}, \quad (i = 1, 2, ..., n) \tag{7.3}$$

Note that (7.3) is essentially the cumulative distribution function of a Student's t distribution and is easy to evaluate.

8. *Stratified Sampling*

It suffices to extend the MML method above to stratified random sampling. The robust method extends along similar lines, and the classical method based on strata means and variances is well known.

Suppose that a population is divided into L non-overlapping strata and there are N_h elements in the hth strata. Let $(h = 1, 2, ..., L)$

$$y_{h1}, y_{h2}, ..., y_{hn_h} \tag{8.1}$$

be a simple random sample of size n_h from the hth strata. The n_h order statistics of this simple random sample are used in equations (3.16) - (3.18) to obtain the estimators $\hat{\mu}_h$ and $\hat{\sigma}_h$ $(h = 1, 2, ..., L)$. These equations also use the information about the shape parameter p in (2.1). In the hth stratum, let p have the value p_h $(h = 1, 2, ..., L)$. In particular, p_h could be all equal.

The MML estimator of the population mean

$$\overline{Y}_N = \frac{1}{N} \sum_{h=1}^{L} N_h \overline{Y}_h, \; N = \sum_{h=1}^{L} N_h \tag{8.2}$$

is

$$\hat{\mu}_{st} = \frac{1}{N} \sum_{h=1}^{L} N_h \hat{\mu}_h \tag{8.3}$$

The standard error of this estimate is given by

$$\pm \sqrt{\left\{ \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \frac{\left(p_h - \frac{3}{2}\right)(p_h + 1)}{n_h p_h \left(p_h - \frac{1}{2}\right)} \frac{1}{n_h} \left(1 - \frac{n_h}{N_h}\right) \hat{\sigma}_h^2 \right\}} \tag{8.4}$$

For $p_h = \infty$ $(h = 1, 2, ..., L)$, $\hat{\mu}_{st}$ reduces to the classical estimator

$$\overline{y}_{st} = \frac{1}{N} \sum_{h=1}^{L} N_h \overline{y}_h \tag{8.5}$$

and (8.4) reduces to the standard error of $\bar{y}_{st}$, i.e.,

$$\pm\sqrt{\left\{\frac{1}{N^2}\sum_{h=1}^{L} N_h^2 \frac{1}{n_h}\left(1-\frac{n_h}{N_h}\right)s_h^2\right\}} \tag{8.6}$$

where

$$\bar{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi} \quad \text{and} \quad s_h^2 = \frac{1}{n_h - 1}\sum_{i=1}^{n_h}(y_{hi} - \bar{y}_h)^2 \tag{8.7}$$

Note that (8.4) is, on the average, considerably smaller than (8.6) except when $p_h = \infty$ $(h = 1, 2, \ldots, L)$ in which case they are exactly the same.

9. *Cost Function*

The simplest cost function is of the form (Cochran [6], pp. 96)

$$C^* = C_0 + \sum_{h=1}^{L} C_h n_h \tag{9.1}$$

For simplicity, assume that $p_h = p$ for all $h = 1, 2, \ldots, L$, i.e., the underlying distribution in all the L strata are identical (other than the location and scale parameters) and are given by (2.1) with a common p. The location and scale parameters in the hth stratum are denoted by μ_h and σ_h, respectively.

Let D be a pre-determined value of the MSE of $\bar{y}_{st}$. The value of n that minimizes the cost C^* is given by (Cochran [6], pp. 96-98).

$$n = \frac{(\Sigma_h W_h \sigma_h \sqrt{C_h})\,\Sigma_h (W_h \sigma_h / \sqrt{C_h})}{D + (1/N)\,\Sigma_h W_h \sigma_h^2}, \quad (W_h = N_h / N) \tag{9.2}$$

Using the estimator $\hat{\mu}_{st}$ and its MSE, and proceeding exactly on the same lines, the value of n which minimizes the cost C^* is given by

$$n = \frac{(\Sigma_h W_h \sigma_h \sqrt{C_h})\,\Sigma_h (W_h \sigma_h / \sqrt{C_h})}{\left\{p\left(p-\frac{1}{2}\right)/\left(p-\frac{3}{2}\right)(p+1)\right\} D + (1/N)\,\Sigma_h W_h \sigma_h^2} \tag{9.3}$$

Since $p(p-\frac{1}{2})/(p-\frac{3}{2})(p+1) > 1$, (8.10) is smaller than (8.9). The use of $\hat{\mu}_{st}$, therefore, leads to substantial savings in the cost.

In continuation to this paper, we are working out results for skew distributions. We will report these results in a future paper.

Concluding remark : From the results presented in this paper it is clear that the utilization of modern statistical inference procedures in sample survey can give considerably more precise estimates than those based on the classical assumption of normality which hardly every holds true; see also Wong [29] who reflects on the divergence of sample survey theory and practice and makes recommendations for closing the gap.

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APPENDIX

Exact values of the variances and covariances :
(1) $(1/\sigma^2)\, V(\hat{\mu}_c)$, (2) $(1/\sigma^2)\, Cov(\hat{\mu}_c, \bar{y}_n)$, (3) $(1/\sigma^2)\, V(\hat{\mu}_n)$, (4) $(1/\sigma^2)\, Cov(\hat{\mu}_n, \bar{y}_n)$,
for the family (2.1)

n	m	p	(1)	(2)	(3)	(4)
4	3.5167	1.5	0.32473	0.37500	0.32472	0.37500
		2.0	0.15804	0.15501	0.15924	0.16882
		2.5	0.20767	0.19286	0.20834	0.21761
		3.5	0.24555	0.21853	0.23569	0.23973
		4.5	0.26117	0.22835	0.24298	0.24509
		5.0	0.23128	0.23999	0.24470	0.24631
		∞	0.28436	0.25000	0.25000	0.25000
5	4.5627	1.5	0.26151	0.31507	0.23909	0.29137
		2.0	0.12377	0.12717	0.12014	0.12233
		2.5	0.16091	0.15689	0.15993	0.16657
		3.5	0.18857	0.17658	0.18519	0.18879
		4.5	0.19976	0.18398	0.19252	0.19452
		5.0	0.20317	0.18617	0.19431	0.19587
		∞	0.21917	0.20000	0.20000	0.20000
6	5.5941	1.5	0.22490	0.27526	0.18782	0.23378
		2.0	0.10355	0.10863	0.09700	0.09560
		2.5	0.13319	0.13293	0.12985	0.13443
		3.5	0.15327	0.14862	0.15235	0.15539
		4.5	0.16316	0.15441	0.15928	0.16106
		5.0	0.16183	0.15611	0.16101	0.16242
		∞	0.17876	0.16667	0.16667	0.16667
7	6.6172	1.5	0.19964	0.24585	0.15402	0.19275
		2.0	0.08975	0.09515	0.08149	0.07847
		2.5	0.11435	0.11561	0.10937	0.11251
		3.5	0.13172	012851	0.12936	0.13188
		4.5	0.13847	0.13318	0.13578	0.13733
		5.0	0.14050	0.13454	0.13741	0.13865
		∞	0.15112	0.14286	0.14286	0.14286

8	7.6352	1.5	0.18065	0.22294	0.13026	0.16261
		2.0	0.07956	0.08481	0.07028	0.06662
		2.5	0.10055	0.10242	0.09451	0.09666
		3.5	0.11501	0.11328	0.11237	0.11448
		4.5	0.12054	0.11714	0.11829	0.11964
		5.0	0.12219	0.11826	0.11982	0.12091
		∞	0.13097	0.12500	0.12500	0.12500
9	8.6497	1.5	0.16562	0.20437	0.11274	0.13971
		2.0	0.07164	0.07660	0.06178	0.05794
		2.5	0.08991	0.09201	0.08324	0.08471
		3.5	0.10223	0.10133	0.09933	0.10109
		4.5	0.10687	0.10460	0.10479	0.10596
		5.0	0.10823	0.10553	0.10621	0.10716
		∞	0.11561	0.11111	0.11111	0.11111
10	9.6617	1.5	0.15333	0.18902	0.09932	0.12196
		2.0	0.06529	0.06991	0.05511	0.05133
		2.5	0.08143	0.08358	0.07438	0.07538
		3.5	0.09211	0.09169	0.08900	0.09048
		4.5	0.09606	0.09450	0.09405	0.09507
		5.0	0.09722	0.09530	0.09537	0.09621
		∞	0.10350	0.10000	0.10000	0.10000
11	10.6719	1.5	0.14302	0.17606	0.08875	0.10788
		2.0	0.06004	0.06434	0.04973	0.04610
		2.5	0.07448	0.07660	0.06723	0.06789
		3.5	0.08386	0.08375	0.08061	0.08188
		4.5	0.08728	0.08620	0.08530	0.08619
		5.0	0.08829	0.08689	0.08654	0.08728
		∞	0.09370	0.09091	0.09091	0.09091
12	11.6807	1.5	0.13422	0.16495	0.08019	0.09649
		2.0	0.05562	0.05962	0.04530	0.04188
		2.5	0.06866	0.07071	0.06134	0.06176
		3.5	0.07699	0.07708	0.07367	0.07475
		4.5	0.08002	0.07924	0.07804	0.07883
		5.0	0.08089	0.07985	0.07920	0.07986
		∞	0.08561	0.08333	0.08333	0.08333

13	12.6883	1.5	0.12660	0.15529	0.07314	0.08713
		2.0	0.05186	0.05557	0.04139	0.03839
		2.5	0.06373	0.06568	0.05640	0.05664
		3.5	0.07122	0.07142	0.06784	0.06877
		4.5	0.07389	0.07333	0.07191	0.07261
		5.0	0.07466	0.07387	0.07300	0.07360
		∞	0.07881	0.07692	0.07692	0.07692
14	13.6951	1.5	0.11009	0.14162	0.06707	0.08041
		2.0	0.04860	0.05206	0.03843	0.03545
		2.5	0.05949	0.06134	0.05220	0.05232
		3.5	0.06626	0.06654	0.06285	0.06367
		4.5	0.06865	0.06826	0.06668	0.06731
		5.0	0.06933	0.06873	0.06770	0.06823
		∞	0.07302	0.07143	0.07143	0.07143
15	14.2534	1.5	0.08500	0.10709	0.06225	0.07275
		2.0	0.03350	0.04103	0.03548	0.03249
		2.5	0.05195	0.05339	0.04858	0.04861
		3.5	0.06029	0.05970	0.05856	0.05929
		4.5	0.06353	0.06199	0.06216	0.06272
		5.0	0.06450	0.06265	0.06313	0.06361
		∞	0.07016	0.06667	0.06667	0.06667
16	15.2704	1.5	0.08123	0.10227	0.05787	0.06708
		2.0	0.03808	0.03713	0.03335	0.03078
		2.5	0.04893	0.04635	0.04544	0.04540
		3.5	0.05656	0.05247	0.05480	0.05543
		4.5	0.05951	0.05475	0.05821	0.05872
		5.0	0.06038	0.05542	0.05913	0.05956
		∞	0.06549	0.06250	0.06250	0.06250
17	16.2857	1.5	0.07787	0.09809	0.05389	0.06252
		2.0	0.03614	0.03258	0.03128	0.02889
		2.5	0.04625	0.04083	0.04267	0.04258
		3.5	0.05329	0.04640	0.05151	0.05206
		4.5	0.05595	0.04848	0.05471	0.05517
		5.0	0.05677	0.04913	0.05561	0.05600
		∞	0.06140	0.05882	0.05882	0.05882

18	17.2994	1.5	0.07483	0.09402	0.05082	0.05794
		2.0	0.03442	0.02884	0.02946	0.02722
		2.5	0.04388	0.03616	0.04168	0.04443
		3.5	0.04959	0.04037	0.04774	0.04828
		4.5	0.05286	0.04305	0.05165	0.05206
		5.0	0.05318	0.04413	0.05248	0.05284
		∞	0.05781	0.05556	0.05556	0.05556
19	18.3120	1.5	0.07203	0.09042	0.04789	0.05426
		2.0	0.03286	0.03557	0.02783	0.02574
		2.5	0.04175	0.04317	0.03804	0.03788
		3.5	0.04780	0.04781	0.04598	0.04642
		4.5	0.04960	0.04885	0.04811	0.04864
		5.0	0.05075	0.04990	0.04969	0.05002
		∞	0.05461	0.05263	0.05263	0.05263
20	19.3234	1.5	0.06954	0.08715	0.04528	0.05098
		2.0	0.03145	0.03405	0.02637	0.02442
		2.5	0.03982	0.04122	0.03608	0.03591
		3.5	0.04548	0.04556	0.04364	0.04403
		4.5	0.04758	0.04706	0.04642	0.04677
		5.0	0.04820	0.04749	0.04718	0.04748
		∞	0.05175	0.05000	0.05000	0.05000